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Algebra

Q.1. ~~Every~~ Prove that every subgroup of a cyclic group is cyclic.

Proof: Let $G = \{a\}$ be a cyclic group generated by a. If $H = G$ or $\{e\}$, then obviously H is cyclic.

Now, let H is a proper subgroup of G.

H contains the element of integral power of a. If $a^x \in H \Rightarrow a^{-1} \in H$. Therefore, H contain elements which are positive as well as negative integral of a. Let k be the least positive integer such that $a^k \in H$. We have to prove that $H = \{a^m\}$.

Now, let $a^t \in H$. Then, by division algorithm Then exist $q + r \in Z$ such that

$$t = kq + r, \quad 0 \leq r < k.$$

Now, $a^k \in H \Rightarrow (a^k)^q \in H \Rightarrow a^{kq} \in H \Rightarrow (a^{kq})^{-1} \in H \Rightarrow a^{-kq} \in H$.

Also, $a^t \in H$, $a^{-kq} \in H \Rightarrow a^t a^{-kq} \in H \Rightarrow a^{t-kq} \in H \Rightarrow a^r \in H$

$\because$ k is the least positive integer such that $a^k \in H$ and $0 \leq r < k$.

$\Rightarrow$ r must be equal to 0

$\Rightarrow t = kq$.

$\therefore a^t = a^{kq} = (a^k)^q$

$\Rightarrow$ every element $a^t \in H$ is of the form $(a^k)^q$

$\Rightarrow$ H is cyclic with generator a^m

Proved

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Algebra

Q.2- prove that a finite group of order n containing an element of order n must be cyclic.

Proof: Let G be a finite group of order n and let $a \in G$, such that $o(a) = n$. Then $o(G) = n = o(a)$.

We have to prove that G is cyclic.

Let H be a cyclic group generated by a, then

$o(H) = o(a) = n$

[$\because$ order of a cyclic group is equal to the order of its generator]

$\Rightarrow$ H can be expressed as

$H = \{a^r : r = 1, 2, 3, \cdots n\}$.

Since, G is a group

then $\Rightarrow a \in G \Rightarrow a^r \in G$ for every integral value of r.

Thus $H \subseteq G$

Moreover, $o(G) = n = o(H)$

$\therefore H = G$, but H is cyclic $\Rightarrow$ G is cyclic.

Q.3- Define cyclic Groups.

Ans: Cyclic Groups:- If a group G contains an element a such that every element $x \in G$ is of the form a^m, where $m \in \mathbb{Z}$, then G is said to be cyclic group G is generated by a i.e. a is the generator of G and we write $G = (a)$.

For example. The multiplication group $G = \{1, -1, i, -i\}$ is a cyclic group with generator i, $i^1 = i$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. G is cyclic, generated by i.